Mathematics II — 09/01/2025 — Part I

1 a) $1/5, -1,$

$2, 1$

b) $Q(x,y) = -6xy + 2y^2$

c) $D_f = \{(x,y) \in \mathbb{R}^2 : y \neq 1 \wedge x \neq 0 \wedge y > 0\}$

d) $\{(1,0), (0,0), (-1,0)\}$

e) $\partial A = \{(x,y) \in \mathbb{R}^2 : x^2 + (y+2)^2 = 4\}$

closed

f) $\lim_{n \to +\infty} f(x_n) = -6$

g) Weierstrass, bounded

h) 9, 2

i) $\frac{\partial f}{\partial y}(x,y) = x \cdot g(x^2, 4x^2)$

j) $f(x,y) = 4x + \cos(x^2 + y)$

k) $H_f(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ global

l) $P_{f,2,(0,0)}(h_1,h_2) = 1 - 3h_2 + \frac{9}{2} h_2^2$

m) $\mathcal{L}(x,y,z,\lambda) = x - 2y + 2z - \lambda(1 - x^2 - y^2 - z^2)$

n) $\displaystyle\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} 1 \, dy \, dx = 2\pi$

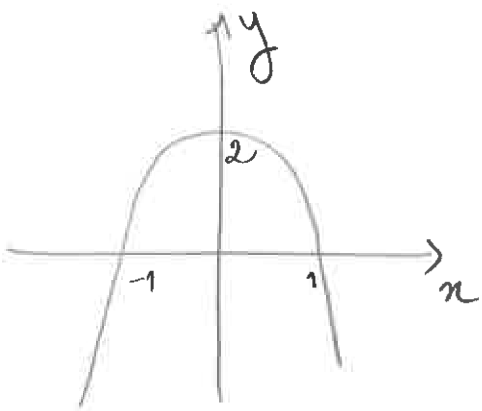
o) $\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 1 \\ y'(0) = -1 \end{cases}$

p) increasing ; 5

q) $y''(x) = -4$

$y'(x) = -4x$

$y(x) = -2x^2 + 2$



Mathematics II - 09/01/2025 - Part II

1 a)

$$A = \begin{pmatrix} \alpha & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & \alpha \end{pmatrix}$$

Using leading minors for $\alpha \in \mathbb{R} \setminus \{-4, 4\}$.

$\Delta_1 = \alpha$

$\Delta_2 = \alpha$

$\Delta_3 = \alpha^2 - 16 \neq 0$ (for $\alpha \in \mathbb{R} \setminus \{-4, 4\}$)

α	$-\infty$	-4		0		4	$+\infty$
Δ_1	$-$	$-$	$-$	0	$+$	$+$	$+$
Δ_2	$-$	$-$	$-$	0	$+$	$+$	$+$
Δ_3	$+$	0	$-$	$-$	$-$	0	$+$
Φ	UND		UND		UND		P.D

$\therefore$

$$\boxed{\begin{array}{l} \alpha > 4 \longrightarrow \Phi \text{ is P.D} \\ \alpha < 4 \wedge \alpha \neq -4 \rightarrow \Phi \text{ is UND.} \end{array}}$$

Using Eigenvalues

$\alpha = 4$ $\qquad A = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & 4 \end{pmatrix}$

$$P(\lambda) = \det \begin{pmatrix} 4-\lambda & 0 & 4 \\ 0 & 1-\lambda & 0 \\ 4 & 0 & 4-\lambda \end{pmatrix} = (4-\lambda)^2(1-\lambda) - 16(1-\lambda)$$

$$= (1-\lambda)\left((4-\lambda)^2 - 16\right)$$

$$= (1-\lambda)(-8\lambda + \lambda^2)$$

$$= (1-\lambda)\,\lambda\,(\lambda - 8)$$

Eigenvalues: $1, 8, 0 \longrightarrow$ Q is S.P.D ($\alpha = 4$)

$\alpha = -4$ $\qquad A = \begin{pmatrix} -4 & 0 & 4 \\ 0 & 1 & 0 \\ 4 & 0 & -4 \end{pmatrix}$

$$P(\lambda) = \det \begin{pmatrix} -4-\lambda & 0 & 4 \\ 0 & 1-\lambda & 0 \\ 4 & 0 & -4-\lambda \end{pmatrix} = (-4-\lambda)^2(1-\lambda) - 16(1-\lambda)$$

$$= (1-\lambda)\left[(+4+\lambda)^2 - 16\right]$$

$$= (1-\lambda)\left[8\lambda + \lambda^2\right]$$

$$= (1-\lambda)(8+\lambda)\,\lambda$$

Eigenvalues: $1, -8, 0 \longrightarrow$ Q is UND. ($\alpha = -4$)

2a) Directional limit $y = m\sqrt{x}, m \in \mathbb{R}$

$$\lim_{x\to 0} \frac{x^2 + m^2 \cdot x}{x} = \lim_{x\to 0} x + m^2 = m^2$$

depends on $\underline{m}$.

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\rightarrow f$ is discontinuous at $(0,0)$

b) f is discontinuous at $(0,0)$

$\Downarrow$ result of the course.

f is not differentiable at $(0,0)$

c) f is positively ~~homogeneous~~ of degree 1. $(x \neq 0)$

$$f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda x} = \lambda\left(\frac{x^2+y^2}{x}\right)$$

$$= \lambda^1 f(x,y)$$

Using Euler Formula, we have

$$x \frac{\partial f}{\partial x}(x,y) + y \frac{\partial f}{\partial y}(x,y) = 1 f(x,y)$$

$\Downarrow$

$x=2$
$y=4$

$$2 \cdot \frac{\partial f}{\partial x}(2,4) + 4\,\frac{\partial f}{\partial y}(2,4) = f(2,4) = \frac{2^2+4^2}{2}$$

$$= 10$$

3a).

$$f(x,y) = y^2 - 2y - x^2 y + 2x^2$$

$$\nabla f(x,y) = (-2xy + 4x;\ 2y - 2 - x^2)$$

$$\nabla f(x,y) = \bar{0} \Leftrightarrow \begin{cases} -2xy + 4x = 0 \\ 2y - 2 - x^2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = 0 \\ 2y - 2 = 0 \end{cases} \vee \begin{cases} -2y + 4 = 0 \\ 2 - x^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \vee \begin{cases} y = 2 \\ x = \pm\sqrt{2} \end{cases}$$

Critical points: $\boxed{(0,1),\ (+\sqrt{2},2),\ (-\sqrt{2},2)}$

$$Hf(x,y) = \begin{pmatrix} -2y+4 & -2x \\ -2x & 2 \end{pmatrix}$$

$$Hf(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$ $\Delta_1 = 2$, $\Delta_2 = 4$ P.D $\Rightarrow$ $(0,1)$ local minimiter

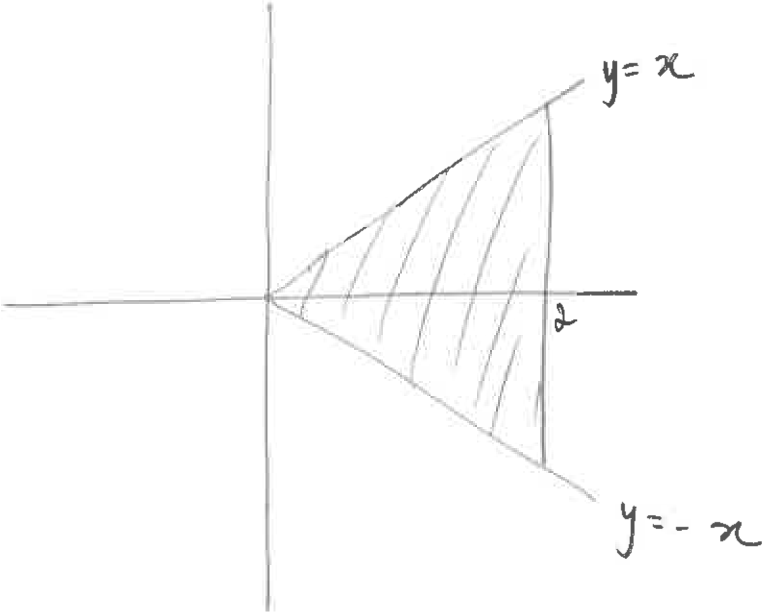
$$Hf(+\sqrt{2},2) = \begin{pmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{pmatrix}$$ $\left.\begin{array}{l}\Delta_1 = 0 \\ \Delta_2 \neq 0\end{array}\right\} \Rightarrow (\sqrt{2},2)$ is a saddle

$$Hf(-\sqrt{2},2) = \begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{pmatrix}$$ $\left.\begin{array}{l}\Delta_1 = 0 \\ \Delta_2 \neq 0\end{array}\right\} \Rightarrow (-\sqrt{2},2)$ is a saddle.

3b) $f(x,0) = -x^2 \cdot (-2) = 2x^2$

$\lim\limits_{x \to +\infty} f(x,0) = +\infty \Longrightarrow f$ is unbounded.

4.)



$$\int_0^2 \int_{-x}^{x} x^2 e^{yx-x^2}\, dy\, dx =$$

$$= \int_0^2 \left[x^2 \cdot \frac{e^{yx-x^2}}{x} \right]_{y=-x}^{x} dx =$$

$$= \int_0^2 \left[x e^0 - x e^{-2x^2} \right] dx$$

$$= \left[\frac{x^2}{2} + \frac{e^{-2x^2}}{4} \right]_0^2 =$$

$$= 2 + \frac{e^{-8}}{4} - 0 - \frac{1}{4} = \frac{7}{4} + \frac{e^{-8}}{4}$$

5 a)

Substituting y for ± 2 in the diff. equation, we get a true equality.

$y_1(x) = 2$ $\qquad y_1'(x) = 0$

$$y' \cdot y = xy^2 - 4x$$
$$\underbrace{0 \cdot 2}_{0} = \underbrace{x \cdot 4 - 4x}_{0} \quad //$$

$\therefore$ y_1 and y_2 are solutions of the diff. equation

b)

$$y' \cdot y = xy^2 - 4x$$
$$y' \cdot y = x\,(y^2 - 4)$$

$$\overset{y \neq \pm 2}{\Leftrightarrow} \frac{y}{y^2-4}\, y' = x$$

$$\Leftrightarrow \frac{1}{2} \ln (y^2 - 4) = \frac{x^2}{2} + c^*, \quad c^* \in \mathbb{R}$$

$$\Leftrightarrow \ln (y^2 - 4) = x^2 + c, \quad c \in \mathbb{R}$$

$$y^2 - 4 = e^{x^2} \cdot k \quad , k \in \mathbb{R}^+$$

$$y^2 = 4 + k e^{x^2}$$

$$\boxed{y_{(x)} = \pm\sqrt{4 + k e^{x^2}}} \quad k \in \mathbb{R}^+$$

$$y(1) = \sqrt{3} \Leftrightarrow \quad \sqrt{3} = \oplus\sqrt{4 + ke}$$

$$4 + ke = 3 \Leftrightarrow ke = -1$$

$$\Rightarrow k = -1/e$$

$$y(x) = \sqrt{4 - e^{x^2-1}}$$

$$4 - e^{x^2-1} > 0$$

$$D_y = \left] -\sqrt{\ln 4 + 1} \, ; \, +\sqrt{\ln 4 + 1} \right[$$

$$-e^{x^2-1} > -4$$

$$e^{x^2-1} < 4$$

$$x^2 - 1 < \ln 4$$

$$x^2 < \ln 4 + 1$$